

Algebraic Approach to the Duality Symmetry

Igor Salom*

Institute of Physics, P.O. Box 57, 11001 Belgrade, Serbia and Montenegro

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Abstract

The duality symmetry of free electromagnetic field is analyzed within an algebraic approach. To this end, the conformal $c(1, 3)$ algebra generators are expressed as operators quadratic in some abstract operators κ^α and π_β which satisfy Heisenberg algebra relations. It is then shown that the duality generator can also be expressed in this manner. Standard issues regarding duality are considered in such a framework. It is shown that duality generator also generates chiral transformations, and the conflict between duality and manifest Lorentz symmetry is analyzed from the viewpoint of symmetry group greater than conformal, in which duality generator appears as a natural part of an $su(2)$ subalgebra.

*Electronic address: isalom@phy.bg.ac.yu

I. INTRODUCTION

Interest in the issue of duality symmetry has significantly increased in recent years, due to its possible application in non-perturbative regimes of various theories, in particular due to its relation with string theories [1, 2]. Naturally, this has increased interest also in the first of the dualities observed – that of free field electric-magnetic duality. This symmetry is interesting in its own right, not only because it deals with something so common as propagation of light, but also because, in spite of simplicity of its form, some aspects of electric-magnetic duality still puzzle the physicists. First of all, the manifest Lorentz covariance seems to be somehow incompatible with electric-magnetic duality: standard covariant Lagrangian of the free electromagnetic field does not possess duality symmetry, as it changes sign upon transformations $E \rightarrow -B$, $B \rightarrow E$. Usual attempts to maintain Lorentz invariance together with the duality symmetry resorted either to construction of non-polynomial action [3] or required an infinite set of fields [4]. Following the idea of Majorana to write a Dirac-like motion equation for photon where no non-physical (gauge) freedom degrees would appear [5], authors [6] proposed a formulation where duality invariance is obtained at the cost of introduction of auxiliary conjugate E and B fields and loss of manifest Lorentz covariance. Even with abandoning of the manifest Lorentz covariance of action, there remains a problem with the generator of (continuous) duality transformations: such generator is either non-local [7] or requires introduction of auxiliary potential [8]. Another curious property of the duality symmetry, revealed in attempts to formulate fermion-like formulation for the electromagnetic field, is its connection with the chiral symmetry [6].

Unlike the analyses of the electromagnetic duality mentioned above that start from Lagrangian formalism, Hamiltonian formalism, or directly from equations of motion, in this paper we attempt to give a kind of unified coverage of these topics from another perspective, namely from algebraic approach. We exploit the fact that conformal algebra (and thus also its Poincare subalgebra) is contained as a subalgebra in algebra of operators quadratic in some abstract operators κ^α and π_β which satisfy Heisenberg algebra relations. In this context, we show that the helicity operator can also be constructed as a quadratic operator in terms of κ^α and π_α . In other words, the helicity operator here appears on the same footing as the conformal operators. We show that this operator generates chiral transformations in the case of helicity $\pm\frac{1}{2}$ states and electromagnetic duality transformations in the case of helicity ± 1 states. The duality symmetry and the problem of its incompatibility with manifest Lorentz symmetry are then considered from a viewpoint of a larger mathematical symmetry that appears in this formulation.

The paper is organized in the following way: in section II we construct conformal algebra in four dimensions $C(1, 3)$ in terms of the Heisenberg algebra operators. In section III we construct single particle Hilbert space and analyze action of the helicity operator in subspaces of helicity $\pm\frac{1}{2}$ and ± 1 . In section IV the issue of duality is considered from the viewpoint of the larger symmetry group which embeds both conformal group and duality transformation. Also, a possible physical interpretation of this larger symmetry is considered. Finally, section V summarizes the results.

Throughout the text, Latin indices i, j, k, \dots will take values 1, 2 and 3, Greek indices from the beginning of alphabet α, β, \dots will take values from 1 to 4 and will in general denote Dirac-like spinor indices, while Greek indices from the middle of alphabet μ, ν, \dots will take values from 0 to 3, denoting Lorentz four-vector indices.

II. CONSTRUCTING POINCARE ALGEBRA FROM HEISENBERG OPERATORS

Let operators κ^α and π_α satisfy Heisenberg algebra in four dimensions: $[\kappa^\alpha, \pi_\beta] = i\delta_\beta^\alpha$, $[\kappa^\alpha, \kappa^\beta] = [\pi_\alpha, \pi_\beta] = 0$.¹ There are three types of quadratic combinations of these operators: quadratic in κ^α , quadratic in π_α and mixed. Hermitian operators of each of these kinds can be written in matrix notation, respectively as:

$$\begin{aligned}\widehat{(A)}_{\kappa\kappa} &\equiv A_{\alpha\beta}\kappa^\alpha\kappa^\beta, \\ \widehat{(A)}_{\pi\pi} &\equiv A^{\alpha\beta}\pi_\alpha\pi_\beta, \\ \widehat{(A)}_{\pi\kappa} &\equiv A^\alpha_{\beta\frac{1}{2}}\{\pi_\alpha, \kappa^\beta\},\end{aligned}\tag{1}$$

where A is an arbitrary four by four real matrix.² However, due to commutativity of operators κ^α among themselves, and π_α among themselves, matrices appearing in definitions $\widehat{(A)}_{\kappa\kappa}$ and $\widehat{(A)}_{\pi\pi}$ are implied to be symmetric.

Such quadratic operators form an algebra with commutation relations easily derived from the Heisenberg algebra relations:

$$\begin{aligned}[\widehat{(A)}_{\pi\kappa}, \widehat{(B)}_{\pi\kappa}] &= i([\widehat{A}, \widehat{B}])_{\pi\kappa}, \\ [\widehat{(A)}_{\pi\kappa}, \widehat{(B)}_{\pi\pi}] &= i(AB + BA^T)_{\pi\pi}, \\ [\widehat{(A)}_{\pi\kappa}, \widehat{(B)}_{\kappa\kappa}] &= -i(A^T B + BA)_{\kappa\kappa}, \\ [\widehat{(A)}_{\pi\pi}, \widehat{(B)}_{\kappa\kappa}] &= -4i(\widehat{AB})_{\pi\kappa}, \\ [\widehat{(A)}_{\kappa\kappa}, \widehat{(B)}_{\kappa\kappa}] &= [\widehat{(A)}_{\pi\pi}, \widehat{(B)}_{\pi\pi}] = 0.\end{aligned}\tag{2}$$

To reveal the Poincare subalgebra in this structure, first we choose a set of six real matrices σ_i and τ_i , $i, \underline{i} = 1, 2, 3$ (four dimensional analogs of Pauli matrices) satisfying

$$[\sigma_i, \sigma_j] = 2\varepsilon_{ijk}\sigma_k, \quad [\tau_{\underline{i}}, \tau_{\underline{j}}] = 2\varepsilon_{\underline{ijk}}\tau_{\underline{k}}, \quad [\sigma_i, \tau_{\underline{j}}] = 0, \tag{3}$$

as a basis of antisymmetric four by four real matrices³ (we distinct tau indices from sigma indices by underlining the former). However, unlike Pauli matrices, these matrices are anti-hermitian, satisfying $\sigma_i^2 = \tau_{\underline{i}}^2 = -1$. As a basis for symmetric matrices we choose nine matrices $\alpha_{ij} \equiv \tau_i\sigma_j$ and unit matrix denoted as α_0 . In order to establish, later on, connection with standard notation, we state one corresponding representation of Dirac gamma matrices:

$$\gamma_0 = i\tau_{\underline{2}}, \quad \gamma_i = \gamma_0\alpha_{\underline{3}i} = i\tau_{\underline{1}}\sigma_i, \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = i\tau_{\underline{3}}. \tag{4}$$

¹ In spite of this, we stress that these operators do not represent coordinates and momenta. Furthermore, they will turn out to transform like Dirac spinors.

² A hat sign over a matrix will be used to emphasize the difference between the operator obtained from a matrix in the sense of definition (1) and the matrix itself.

³ One possible realization of such matrices is, for example: $\sigma_1 = -i\sigma_y \times \sigma_x$, $\sigma_2 = -iI_2 \times \sigma_y$, $\sigma_3 = -i\sigma_y \times \sigma_z$, $\tau_1 = i\sigma_x \times \sigma_y$, $\tau_2 = -i\sigma_z \times \sigma_y$, $\tau_3 = -i\sigma_y \times I_2$, where σ_x , σ_y and σ_z are standard two dimensional Pauli matrices and I_2 is a two dimensional unit matrix.

Now, set of 36 operators

$$\left\{ \widehat{(\tau_{\underline{i}})}_{\pi\kappa}, \widehat{(\sigma_j)}_{\pi\kappa}, \widehat{(\alpha_0)}_{\pi\kappa}, \widehat{(\alpha_{\underline{i}j})}_{\pi\kappa}, \widehat{(\alpha_0)}_{\pi\pi}, \widehat{(\alpha_{\underline{i}j})}_{\pi\pi}, \widehat{(\alpha_0)}_{\kappa\kappa}, \widehat{(\alpha_{\underline{i}j})}_{\kappa\kappa} \right\} \quad (5)$$

can be chosen as basis of algebra of quadratic operators.

Among the operators from this set, let us discard all those with underlined index having values 1 and 2. This resembles an idea to introduce a “preferred tau direction” (here, for concreteness, this “direction” was taken to be along the third “axis”) and to drop out every entity which has tau indices but is not along this preferred direction. What we are left with is a subalgebra isomorphic with conformal algebra $C(1,3)$ plus one additional generator that commutes with the rest of the subalgebra. Now we introduce new notation for the remaining generators:

$$\begin{aligned} M_{ij} &= \varepsilon_{ijk} J_k \equiv \varepsilon_{ijk} \left(\frac{\widehat{\sigma_k}}{2} \right)_{\pi\kappa}, \\ M_{i0} = -M_{0i} &= N_i \equiv \left(\frac{\widehat{\alpha_{3i}}}{2} \right)_{\pi\kappa}, \quad D \equiv \left(\frac{\widehat{\alpha_0}}{2} \right)_{\pi\kappa}, \\ P_i &\equiv \left(\frac{\widehat{\alpha_{3i}}}{2} \right)_{\pi\pi}, \quad P_0 \equiv \left(\frac{\widehat{\alpha_0}}{2} \right)_{\pi\pi}, \\ K_i &\equiv \left(\frac{\widehat{\alpha_{3i}}}{2} \right)_{\kappa\kappa}, \quad K_0 \equiv -\left(\frac{\widehat{\alpha_0}}{2} \right)_{\kappa\kappa}. \end{aligned} \quad (6)$$

The additional remaining operator is

$$Y_3 \equiv \left(\frac{\widehat{\tau_3}}{2} \right)_{\pi\kappa}, \quad (7)$$

which commutes with all of the conformal generators. The operator Y_3 is in fact the helicity operator, as can be seen from the following mathematical identity:

$$\mathbf{P} \cdot \mathbf{J} = P^0 Y_3. \quad (8)$$

This can be verified most easily by a direct calculation, using some concrete realization of the σ and τ matrices. [Note that $\mathbf{P} \cdot \mathbf{J} = -(P_1 J_1 + P_2 J_2 + P_3 J_3)$.]

We will show that in the subspace of helicity $\pm \frac{1}{2}$ states this identity directly turns into Dirac equation, while in the subspace of helicity ± 1 states the same identity explicitly turns into a pair of Maxwell’s equations. Furthermore, we will demonstrate that the operator Y_3 in the former case generates chiral transformations while in the latter case it generates electromagnetic duality transformations.

III. SINGLE PARTICLE HILBERT SPACE

To this end we start by considering single particle Hilbert space, which is analogue of the Hilbert space of non-relativistic quantum mechanics with operators of coordinates and

momenta replaced by four pairs of $(\kappa^\alpha, \pi_\alpha)$ operators. In this space basis vectors $|p, \theta, \varphi, \mathbf{y}_3\rangle$ exist that are eigenstates of both helicity operator and spatial momentum:

$$\begin{aligned} Y_3 |p, \theta, \varphi, \mathbf{y}_3\rangle &= \mathbf{y}_3 |p, \theta, \varphi, \mathbf{y}_3\rangle, \\ P^1 |p, \theta, \varphi, \mathbf{y}_3\rangle &= p \sin \theta \cos \varphi |p, \theta, \varphi, \mathbf{y}_3\rangle, \\ P^2 |p, \theta, \varphi, \mathbf{y}_3\rangle &= p \sin \theta \sin \varphi |p, \theta, \varphi, \mathbf{y}_3\rangle, \\ P^3 |p, \theta, \varphi, \mathbf{y}_3\rangle &= p \cos \theta |p, \theta, \varphi, \mathbf{y}_3\rangle, \end{aligned}$$

where $p \in [0, \infty)$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$ and $\mathbf{y}_3 = 0, \pm\frac{1}{2}, \pm 1, \dots$. Spherical expressions for momenta eigenvalues are more appropriate here, since detailed calculation reveals that components of wave functions that have half-odd integer values of helicity must be 2π antiperiodic in angle φ when expressed in this basis.

Algebraic identity that is derived from definition (6):

$$\eta^{\mu\nu} P_\mu P_\nu = (P_0)^2 - (P_1)^2 - (P_2)^2 - (P_3)^2 = 0. \quad (9)$$

implies that all states in this Hilbert space must be massless (not surprising due to existence of conformal symmetry), so the value of energy P_0 of these states is simply the magnitude of momentum. Normalization is chosen to provide that under Lorentz transformations the states transform as: $\Lambda |p, \mathbf{y}_3 = 0\rangle = \sqrt{\frac{p'^0}{p^0}} |\Lambda p, \mathbf{y}_3 = 0\rangle$.

Next, we define *scalar field vectors*, namely Hilbert space vectors that correspond to states of a single scalar particle created at a given point x :

$$|\phi(x)\rangle \equiv \int_{\mathcal{R}^3} \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2p^0}} e^{ip_\mu x^\mu} |p, \mathbf{y}_3 = 0\rangle. \quad (10)$$

Such vectors have simple Lorentz transformation properties: $\Lambda |\phi(x)\rangle = |\phi(\Lambda x)\rangle$.⁴

For arbitrary Hilbert state $|f\rangle$ we define its *scalar field representation* as $\phi_f(x) \equiv \langle \phi(x) | f \rangle$. Direct calculation shows that action of conformal generators in this representation (defined for arbitrary generator G as $G\phi_f(x) \equiv \langle \phi(x) | G | f \rangle$) reduce to standard formulas for classical fields. Direct calculation also provides correct value for dilatation charge for scalar field, something that is in the standard approach usually inserted “by hand” to make the theory dilatationally invariant (correct value of dilatation charge is automatically obtained also for other fields, for example spinor and helicity ± 1 fields). The scalar field representation function of arbitrary state defined in such way satisfies Klein-Gordon equation, due to equality (9).

The helicity $\pm\frac{1}{2}$ states are obtained by applying the π_α operators to the scalar states. As these operators transform under the spinor representation of Lorentz group (as can be verified by calculating commutator $[M_{\mu\nu}, \pi_\alpha]$), states

$$|\psi_\alpha(x)\rangle \equiv \sqrt{2} \pi_\alpha |\phi(x)\rangle \quad (11)$$

⁴ Uniqueness of these vectors can be better understood if the vector $|\phi(0)\rangle$ is expressed in basis $|\pi_1, \pi_2, \pi_3, \pi_4\rangle$ of operator π_α eigenstates, where its wave function is simply a constant ($|\phi(0)\rangle \sim \int d^4\pi |\pi_1, \pi_2, \pi_3, \pi_4\rangle$). Action of any κ^α operator on such state vanishes, so it is obviously invariant under action of any operator of form $A^\alpha_\beta \pi_\alpha \kappa^\beta$ including Lorentz generators.

transform like spinors. More precisely, function

$$\psi_{f\alpha}(x) \equiv \langle \psi_\alpha(x) | f \rangle, \quad (12)$$

that we are going to call *spinor field representation* of a state $|f\rangle$, transforms as a classical spinor field, under both Lorentz and conformal group. In particular, we find:

$$\begin{aligned} P_\mu \psi_{f\alpha}(x) &= \partial_\mu \psi_{f\beta}(x), \\ M_{\mu\nu} \psi_{f\alpha}(x) &= i \left((x_\mu \partial_\nu - x_\nu \partial_\mu) \delta_\alpha^\beta + (\sigma_{\mu\nu})_\alpha^\beta \right) \psi_{f\beta}(x) \end{aligned} \quad (13)$$

where, as usually, $\sigma_{\mu\nu} \equiv \frac{1}{4}[\gamma_\mu, \gamma_\nu]$.

We now find action of the \underline{Y}_3 operator in the spinor field representations (in the scalar field case this operator trivially reduces to zero):

$$\begin{aligned} \underline{Y}_3 \psi_{f\alpha}(x) &\equiv \langle \psi_\alpha(x) | \underline{Y}_3 | f \rangle \\ &= i \left(\frac{\tau_3}{2} \right)_\alpha^\beta \psi_{f\beta}(x) = \frac{1}{2} (\gamma_5)_\alpha^\beta \psi_{f\beta}(x). \end{aligned} \quad (14)$$

So, in the spinor field representation operator \underline{Y}_3 effectively turns into the chiral charge matrix γ_5 .

To demonstrate that function $\psi_{f\alpha}(x)$ behaves like massless Dirac field we will show that the Dirac equation is satisfied. Using results (13) and (14) mathematical identity (8) directly leads to massless Dirac equation for spinor field functions:

$$\begin{aligned} 0 &= \langle \psi_\alpha(x) | P_0 \underline{Y}_3 + \sum_i P_i J_i | f \rangle \\ &= \left(\frac{i}{2} (\gamma_5)_\alpha^\beta \partial_0 + \sum_{ijk} i \partial_i \varepsilon_{ijk} \left(i x_j \partial_k \delta_\alpha^\beta + \frac{i}{2} (\sigma_{jk})_\alpha^\beta \right) \right) \psi_{f\beta} \\ &= \left(\frac{i}{2} (\gamma_5)_\alpha^\beta \partial_0 - \sum_i \frac{i}{2} (\gamma_5 \gamma_0 \gamma_i)_\alpha^\beta \partial_i \right) \psi_{f\beta}. \end{aligned} \quad (15)$$

Suppressing the spinorial indices and multiplying by $2\gamma_0\gamma_5$ from the left, we obtain the massless Dirac equation in its standard form:

$$i\gamma^\mu \partial_\mu \psi_f(x) = 0. \quad (16)$$

Just as we applied π_α operators once to scalar field vectors in order to obtain basis for helicity $\pm\frac{1}{2}$ states, we can apply these operators twice, i.e. $\pi_\alpha \pi_\beta |\phi(x)\rangle$ to obtain basis for field representation of helicity ± 1 states. However, four out of ten possible quadratic combinations of $\pi_\alpha \pi_\beta$ will not change helicity – these are ones corresponding to momenta (since momenta commute with \underline{Y}_3). Using the six remaining combinations we define E and B vectors:

$$|E_i(x)\rangle \equiv \widehat{(\alpha_{1i})}_{\pi\pi} |\phi(x)\rangle, \quad (17a)$$

$$|B_i(x)\rangle \equiv -\widehat{(\alpha_{2i})}_{\pi\pi} |\phi(x)\rangle. \quad (17b)$$

Here linear combination $|E_i(x)\rangle \mp i|B_i(x)\rangle$ has helicity value ± 1 .

Corresponding E and B field representation functions:

$$E_{f_i}(x) \equiv \langle E_i(x) | f \rangle, \quad B_{f_i}(x) \equiv \langle B_i(x) | f \rangle, \quad (18)$$

have the same Lorentz transformation properties as electric and magnetic fields, respectively.⁵

For representation of Y_3 operator we find:

$$\begin{aligned} Y_3 E_{f_i}(x) &\equiv \langle E_i(x) | Y_3 | f \rangle = i B_{f_i}(x), \\ Y_3 B_{f_i}(x) &\equiv \langle B_i(x) | Y_3 | f \rangle = -i E_{f_i}(x) \end{aligned} \quad (19)$$

that results in following finite transformations:

$$\begin{aligned} E_{f_i}(x) &\longrightarrow E'_{f_i}(x) = E_{f_i}(x) \cos \phi - B_{f_i}(x) \sin \phi, \\ B_{f_i}(x) &\longrightarrow B'_{f_i}(x) = E_{f_i}(x) \sin \phi + B_{f_i}(x) \cos \phi \end{aligned} \quad (20)$$

corresponding to change of state given by $|f\rangle \longrightarrow \exp(i\phi Y_3)|f\rangle$.

In the similar manner as the Dirac equation was derived from the helicity identity (8) in the spinor field representation, now a pair of Maxwell's equations is derived from the same identity:

$$\begin{aligned} \langle E_i(x) | P_j J_j | f \rangle &= -\langle E_i(x) | P_0 Y_3 | f \rangle \\ \Rightarrow (s_j)_{ik} \partial_j E_{f_k} &= -\partial_0 B_{f_i} \Rightarrow \varepsilon_{ijk} \partial_j E_{f_k} = -\partial_0 B_{f_i}, \\ \langle B_i(x) | P_j J_j | f \rangle &= -\langle B_i(x) | P_0 Y_3 | f \rangle \\ \Rightarrow (s_j)_{ik} \partial_j B_{f_k} &= \partial_0 E_{f_i} \Rightarrow \varepsilon_{ijk} \partial_j B_{f_k} = \partial_0 E_{f_i}. \end{aligned} \quad (21)$$

Summation over repeated indices is implied and matrices s_j are matrices generating rotations in three dimensional vector representation of rotation group. Matrix notation of intermediate results in (21) is the essence of what is sometimes called fermion-like formulation for electromagnetic field [5, 6]. (One can draw closer parallels to Majorana original fermion-like formulation by expressing these results in terms of linear combinations $E_{f_i}(x) \pm i B_{f_i}(x)$ of definite helicity.)

The other two Maxwell equations can be derived from mathematical identity⁶

$$\sum_i \widehat{(\alpha_{ji})}_{\pi\pi} \widehat{(\alpha_{ki})}_{\pi\pi} = \delta_{jk} \widehat{(\alpha_0)}_{\pi\pi}^2, \quad (22)$$

by taking consecutively $j = 3, k = 1$ and $j = 3, k = 2$ (Choice $j = 3, k = 3$ gives identity $P^\mu P_\nu = 0$).

Since the functions $E_{f_i}(x)$ and $B_{f_i}(x)$ satisfy free Maxwell equations, we may recognize transformations (20) as (continuous) duality transformations of free electromagnetic field.

⁵ It is true that so defined E_{f_i} and B_{f_i} functions can take complex values, which is not a property of standard electric and magnetic fields. However, this is hardly avoidable in one first quantization approach like this, where E and B functions are understood to play role of a photon wave function (the idea that physical E and B fields instead of potential A_μ should be related to photon wave function is usually attributed to Majorana [5]).

⁶ Fact that equations of motion are simply mathematical tautologies is a nice characteristic of this approach.

These results can easily be generalized to field representations of arbitrary helicity, and thus equivalent of duality (chirality) can also be defined for these cases. The same is true for generalization of the massless Dirac equation. It is clearly seen here that this equation is essentially just a helicity eigenvalue problem.

To conclude this section we note that in this approach duality generator appears as a well defined operator, a part of the starting algebra, whose action on free E and B fields (20) is localized in space-time. As a matter in fact this operator is also the helicity operator as well as the fermion chiral symmetry generator. The result that helicity operator (in massless case) generates duality transformations essentially agrees with findings of [9], in spite of the differences in approach. The connection of duality and chirality is also not new. Conclusion that “duality is a kind of chirality” can be found in ref. [6]. However, the authors used there the phrase “kind of” since such a conclusion was based simply on the fact that they implemented duality transformations using a matrix that anticommutes with analogues of Dirac γ_μ matrices appearing in a fermion-like formulation for the electromagnetic field. The connection of duality with chiral symmetry is more clearly established with two relations (14) and (19) and in this framework it turns out that “duality is chirality”, i.e. both symmetries are generated by the same operator.

All of these conclusions were derived for the massless and free field case. Nevertheless, let us briefly consider more general case and, in the light of this connection of chirality with duality, consider the following. It is well known that the mass term of the Dirac equation

$$i\gamma^\mu \partial_\mu \Psi(x) = m\Psi(x) \quad (23)$$

breaks chiral invariance, since, for example, a “90° chiral rotation” would require the standard mass term to be replaced with mass term of the form $i\gamma_5 m$:

$$i\gamma^\mu \partial_\mu \Psi(x) = i\gamma_5 m \Psi(x). \quad (24)$$

On the other hand the same symmetry transformation (appearing now as a duality transformation) is expected to turn an electric charge into a magnetic charge. As a consequence of this somewhat simplified consideration a possibility arises that some relation might exist between magnetic charges and $i\gamma_5 m$ type of the mass term.⁷ As the discussion of fields with sources is out of the scope of this paper, we shall not discuss such a possibility in more details.

IV. SYMMETRIES BEYOND CONFORMAL

The algebraic formulation of this work provides a different way to understand duality and also offers a new perspective on conflict between Lorentz covariance and duality symmetry. To fully demonstrate this it is instructive to take a broader viewpoint. Namely, let us recall that conformal algebra, when expressed using Heisenberg operators (6), is from a mathematical point of view natural part of a greater algebra of all quadratic operators in π_α and κ^α (5). If we include the rest of these operators in consideration, it is first

⁷ A similar mass term appears, for example, in papers of Raspini [10] and Dvoeglazov [11], but without the imaginary constant. Lack of this constant in their case introduces exotic massless and tachyonic solutions, which do not occur otherwise.

noticed that duality generator Y_3 mathematically belongs to an $su(2)$ subalgebra generated by $Y_1 \equiv \widehat{(\frac{\tau_1}{2})}_{\pi\kappa}$, $Y_2 \equiv \widehat{(\frac{\tau_2}{2})}_{\pi\kappa}$ and Y_3 . This is an unusual result. This $su(2)$ algebra commuting with rotational generators we designate as *dual spin* algebra. The dual-spin generators and rotation generators, together with nine operators $N_{ij} \equiv \widehat{(\frac{\alpha_{ij}}{2})}_{\pi\kappa}$ [three of these with $i = 3$ are boosts according to (6)] form another algebra [$sl(4, R)$ isomorphic], which is an extension of the Lorentz algebra.

On the other hand four momenta P_μ naturally fit into a set of ten operators quadratic in π_α : P_0 and nine operators $P_{ij} \equiv \widehat{(\frac{\alpha_{ij}}{2})}_{\pi\pi}$. Just as four-momentum transforms under irreducible vector representation of Lorentz group these ten operators belong to an irreducible representation of the extended Lorentz group. Under symmetry reduction from this extended Lorentz group to Lorentz group, this ten dimensional representation decomposes into irreducible representations of Lorentz group as $(\frac{1}{2}, \frac{1}{2}) \oplus (1, 0) \oplus (0, 1)$. Here, components of four momenta P_μ belong to $(\frac{1}{2}, \frac{1}{2})$ subspace, while six operators $P_{11}, P_{12}, P_{13}, P_{21}, P_{22}, P_{23}$ transform under $(1, 0) \oplus (0, 1)$ Lorentz representation, i.e. they transform as an anti-symmetric second rank Lorentz tensor. This makes sense when we repeat our definition of $E_{fi}(x)$ and $B_{fi}(x)$ functions (17, 18), here written as:

$$E_{fi}(x) \equiv 2\langle \phi_f(x) | P_{1i} | f \rangle, \quad B_{fi}(x) \equiv -2\langle \phi_f(x) | P_{2i} | f \rangle. \quad (25)$$

From this aspect duality generator, being the third component of the dual spin, rotates electric field that is “along the first dual axis” into magnetic field that is “along the second dual axis”.⁸ It is clear that $(\mathbf{E})^2 + (\mathbf{B})^2$ is the only dually and rotationally invariant quadratic function of E and B , whereas product $2\mathbf{E} \cdot \mathbf{B}$ and the standard Lagrangian $(\mathbf{E})^2 - (\mathbf{B})^2$ transform into each other. This is completely analogous to the case of, for example, spatial momenta where $p_x^2 + p_y^2$ is invariant under rotations around z axis, whereas under the same rotations functions $p_x^2 - p_y^2$ and $2p_x p_y$ transform into each other. Existence of the two mass terms (23) and (24) can be understood also on the basis of the extended Lorentz group: these mass terms belong to a six dimensional representation of the extended Lorentz group decomposing into $(\frac{1}{2}, \frac{1}{2}) \oplus (0, 0) \oplus (0, 0)$ Lorentz representations where one of the two Lorentz scalars is “along the first dual axis” and the other is “along the second dual axis”. Furthermore, we conclude that if tensor components $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ are to transform in the same way as six entities (E, B) (25) with respect not only to Lorentz but also and to duality transformations, potential A_μ needs additional “underlined” dual spin indices.

Apart from the operators already mention in this section, the full algebra of quadratic Heisenberg operators also includes an extended set of ten “conformal-like” operators K_0 , $K_{ij} \equiv \widehat{(\frac{\alpha_{ij}}{2})}_{\kappa\kappa}$ and one dilatation generator D already defined in (6). This whole, 36 dimensional algebra can be mathematically seen as an extension of conformal algebra and it is isomorphic with symplectic algebra in four dimensions.

A natural question in this context is about the interpretation of the new operators appearing in this extended conformal algebra. So far, this bigger algebra was essentially considered only as a mathematical extension appropriate for this formulation. Now we briefly discuss a possible physical interpretation of the whole algebra as a space-time symmetry.

⁸ If the rotation around the “third dual axis” in positive direction is introduced with a minus sign in exponential $[|f\rangle \rightarrow \exp(-i\phi Y_3)|f\rangle]$ then, strictly speaking, the results (20) and (24) correspond to dual rotations in negative direction.

If this is to be a physical symmetry, it can only be a broken one. It is interesting that a relatively simple form of symmetry breaking can reduce this big symmetry group (with 36 generators) to conformal symmetry (15 generators) multiplied by $U(1)$ group of duality transformations. It is sufficient that dual-spin $SU(2)$ group is strongly broken to its $U(1)$ subgroup, for example by some effective interaction with potential of the form $const \cdot (Y_3)^2$, where the multiplying constant is sufficiently large (let us say of order of Planck mass divided by Planck constant squared). Such a term would effectively reduce low energy physics to the Hilbert space subspace characterized with $y_3 = 0$ while the remaining space-time symmetry would be conformal group. However, symmetry breaking provided by such a simple term is inadequate because of other reasons (for example, non scalar states in this simplest formulation would acquire enormous masses). Nevertheless this possibility incorporates one, to our opinion, appealing idea that fundamental postulated symmetry of space-time fits into some clear mathematical pattern (here it is algebra of all quadratic Heisenberg operators), rather than to be just given by not so simple structural constants (as in the case of Poincare symmetry). It might be argued that the former case contains “less information” and thus that it is favored by Occam’s razor. Due to remote associations with the standard model, it is also potentially interesting that the necessary symmetry breaking is connected with an $SU(2)$ to $U(1)$ breaking, where the group in question is related with chirality. We shall not dwell any longer on this topic, remarking only that if any realistic model is sought with the symmetry given by algebra of operators (2, 5), it seems more appropriate to start with a trilinear generalization of Heisenberg commutation relations of the form $[[\pi_\alpha, \pi_\beta]_+, \pi_\gamma]_- = 0$, $[[\kappa^\alpha, \kappa^\beta]_+, \kappa^\gamma]_- = 0$, $[[\kappa^\alpha, \pi_\beta]_+, \kappa^\gamma]_- = 2i\delta_\beta^\gamma \kappa^\alpha$, $[[\pi_\alpha, \kappa^\beta]_+, \pi_\gamma]_- = 2i\delta_\beta^\gamma \pi_\alpha$, (a graded algebra isomorphic to four dimensional para-Bose algebra whose Green’s representations [12] seem adequate for representing multiparticle states).

At the end, it should be mentioned that understanding the whole algebra as the one generating physical symmetry would have consequences also on the issue of duality. Invariance of Lagrangian terms and matching of transformation properties [for example of $\partial_\mu A_\nu - \partial_\nu A_\mu$ and E, B given by (25)] should be then considered while regarding the full extended conformal group.

V. CONCLUSION

The conformal generators were expressed in this paper as quadratic functions of operators satisfying Heisenberg algebra. In such a formulation it turned out to be possible to express the helicity operator in the same way, putting it on the same level with the conformal algebra generators. This helicity operator upon action on helicity $\pm\frac{1}{2}$ states behaved as chirality generator and upon action on combination of helicity ± 1 states behaved as duality generator. Thus we put the parallels of duality and chirality [6] on stronger grounds. As a convenient feature of this approach we also demonstrated that massless motion equations without sources appear here as mathematical tautologies. Next we pointed out that an unusual perspective on concept of duality is obtained from the viewpoint of complete algebra of quadratic Heisenberg operators, where the duality generator naturally fits into one $su(2)$ algebra. Finally, this large algebra was briefly discussed as a candidate for a physical symmetry of universe. Although the most simple symmetry breaking mechanism mentioned in the section 4 does not meet some of the basic experimental requirements, it was concluded that a simple symmetry breaking of $SU(2)$ dual-spin group to its $U(1)$ duality subgroup could reduce space-time symmetry to conformal symmetry.

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